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An exact solution for electrons in a time-dependent magnetic field

D. Laroze, R. Rivera *

Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4059, Valparaíso, Chile

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Abstract

In this work we study the dynamics of electrons in presence of a uniform time-dependent magnetic field. An exact solution for the wave function time evolution is obtained when the initial state is a superposition of Landau levels. For a given time dependence of the magnetic field, the time-evolved wave function differs from the initial wave function by a dynamic phase factor.
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1. Introduction

The dynamics of electrons in magnetic fields has played a fundamental role in physics and its application to technology in a wide spectrum of topics, such as the Aharonov–Bohm effect [1–4], the bidimensional electron gas at the interface of semiconductor heterostructures [5], cyclotron resonance [6], magnetoplasmon resonance [7], magnetoresistance [8], and electromagnetic lenses with time-dependent magnetic fields [9–12]. In the case of time-dependent quantum systems it appears in addition the phenomenon of dynamic phases [13,14], illustrated for example in Refs. [15,16] in the case of the time-dependent harmonic oscillator. All these situations are interesting in themselves, and present rather complex mathematical structures; therefore exact analytic solutions have been found only in a few special cases.

In the present work, we study the dynamics of electrons in the presence of a uniform time-dependent magnetic field, and we find an exact solution for the corresponding propagator of the Schrödinger equation. An analytical expression for the time-evolved wave function is found when the initial state is a superposition of Landau levels; in particular it is shown that a dynamical phase appears in the time evolution of the wave function as a consequence of the time-dependence of the magnetic field.

2. Theoretical model

In this section we will develop the quantum mechanical formalism to describe the dynamics of electrons in a uniform time-dependent magnetic field, $\mathbf{B}(t)$. Since the corresponding Hamiltonian is time-dependent, energy will not be conserved and there will not be an energy spectrum. Therefore, the problem must be approached by directly solving the time-dependent Schrödinger equation:

$$\frac{1}{2m}\left[\mathbf{P}+\frac{e}{c}\mathbf{A}\right]^2\Phi(\mathbf{r},t)=i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{r},t), \tag{1}$$

where m is the electron mass, $q=-e$ is the electron charge, c is the speed of light and the vector potential is chosen to be $\mathbf{A}=-\mathbf{r}\times\mathbf{B}/2$. The temporal evolution of the system from an initial instant t_0 to a final instant t_f is given by

$$\Phi(\mathbf{r}_f,t_f)=\int d^3r_i\,G(\mathbf{r}_f,t_f,\mathbf{r}_i,t_0)\Phi_0(\mathbf{r}_i,t_0), \tag{2}$$

where $G(\mathbf{r}_f,t_f,\mathbf{r}_i,t_0)$ is the propagator and $\Phi_0(\mathbf{r}_i,t_0)$ is the initial wave function. If the magnetic field has the form $\mathbf{B}(t)=$

* Corresponding author. Tel.: +56 32 273136; fax: +56 32 273529.
E-mail addresses: david.laroze@usm.cl (D. Laroze), rrivera1@vtr.net (R. Rivera).

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$B(t)\hat{\mathbf{z}}$, then the electron dynamics decomposes in a free motion in the z direction and a transversal evolution to be studied. Hence, the wave function may be written as $\Phi(\mathbf{r},t) = \psi(\mathbf{r}_\perp,t)\chi(z,t)$ where $\chi(z,t)$ is the wave function associated to the free motion and $\psi(\mathbf{r}_\perp,t)$ is the transversal wave function. The propagator can be cast in the form

$$G(\mathbf{r}_f,t_f,\mathbf{r}_i,t_0) = g(z_f,t_f,z_i,t_0)G_\perp(\mathbf{r}_{f\perp},t_f,\mathbf{r}_{i\perp},t_0), \tag{3}$$

where $g(z_f,t_f,z_i,t_0)$ is the free propagator in the z direction [17]:

$$g(z_f,t_f,z_i,t_0) = \left(\frac{m}{2\pi i\hbar(t_f-t_0)}\right)^{1/2} \exp\left\{\frac{im(z_f-z_i)^2}{2\hbar(t_f-t_0)}\right\} \tag{4}$$

and where $G_\perp(\mathbf{r}_{f\perp},t_f,\mathbf{r}_{i\perp},t_0)$ is the transversal propagator [10]

$$G_\perp(\mathbf{r}_{f\perp},t_f,\mathbf{r}_{i\perp},t_0) = \frac{m}{2\pi i\hbar\mu_1(t_f)} \exp\left[\frac{im}{2\hbar\mu_1(t_f)}\left[\dot{\mu}_1(t_f)\mathbf{r}_{f\perp}^2 + \mu_2(t_f)\mathbf{r}_{i\perp}^2 - 2\mathbf{r}_{f\perp}\cdot\mathbb{R}\big(\theta(t_f,t_0)\big)\mathbf{r}_{i\perp}\right]\right], \tag{5}$$

where

$$\mathbb{R}\big(\theta(t,t_0)\big) = \begin{bmatrix} \cos\theta(t,t_0) & -\sin\theta(t,t_0) \\ \sin\theta(t,t_0) & \cos\theta(t,t_0) \end{bmatrix} \tag{6}$$

with $\theta(t,t_0) = \int_{t_0}^{t}\omega(t')\,dt'$; and where $\mu_{1,2}(t)$ denote two linear-independent solutions of the equation

$$\ddot{\mu}(t) + \omega(t)^2\mu(t) = 0 \tag{7}$$

with $\omega(t) = eB(t)/(2mc)$. It is convenient to choose $\mu_1(t)$ having dimensions of time and $\mu_2(t)$ being dimensionless, and to choose the initial conditions $\mu_1(t_0) = \dot{\mu}_2(t_0) = 0$, $\mu_2(t_0) = \dot{\mu}_1(t_0) = 1$. When the magnetic field is periodic, Eq. (7) becomes the Hill equation. In general, the proprieties of the system follow from the behavior of the solutions of Eq. (7).

In what follows we will analyze the temporal evolution of the transversal wave function. For this purpose we suppose that for $t < t_0$ the magnetic field is static and of magnitude B_0. The electron wave function at $t = t_0$ can be expressed as a superposition of Landau states:

$$\psi_0(\mathbf{r}_\perp,t_0) = \sum_n C_n\psi_n(\mathbf{r}_\perp,t_0), \tag{8a}$$

where the C_n are constants and

$$\psi_n(\mathbf{r}_\perp,t_0) = \left(\frac{1}{\sqrt{2\pi}}\right)R_n(r)\exp(in\phi), \tag{8b}$$

where r, ϕ are the polar coordinates of $\mathbf{r}_\perp$. The radial function $R_n(r)$ is a solution of the equation [18]:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \left(\beta - 2n\gamma - \frac{n^2}{r^2} - \gamma^2 r^2\right)\right)R_n(r) = 0 \tag{9}$$

with $\beta = -k_z^2 + 2mE/\hbar^2$ and $\gamma = eB_0/2c\hbar$. The normalizability of the wave function requires $n + |n| + 1 - \beta/2\gamma = -2\zeta$, where ζ is a positive integer. The transversal temporal evolution is then described by

$$\psi_n(\mathbf{r}_{f\perp},t_f) = \frac{m}{2\pi i\mu_1(t_f)\hbar}\exp\left(i\frac{m\dot{\mu}_1(t_f)}{2\hbar\mu_1(t_f)}\mathbf{r}_{f\perp}^2\right)\eta_n(\mathbf{r}_{f\perp},t_f,\mathbf{r}_{i\perp},t_0), \tag{10}$$

where use has been made of Eq. (5) and where

$$\eta_n(\mathbf{r}_{f\perp},t_f,\mathbf{r}_{i\perp},t_0) = \int_0^\infty dr_i\,r_i\exp\left[iA(t_f)r_i^2\right]R_n(r_i) \times \int_0^{2\pi} d\phi_i\exp\left[in\phi_i - iK(t_f)r_i\cos\big(\phi_i - \phi_f + \theta(t_f,t_0)\big)\right] \tag{11}$$

with

$$A(t_f) = m\mu_2(t_f)/\big(2\hbar\mu_1(t_f)\big) \tag{12}$$

and

$$K(t_f) = \big[m/\big(\hbar\mu_1(t_f)\big)\big]r_f. \tag{13}$$

The angular integral in Eq. (11) is of the form:

$$I = \int_0^{2\pi} d\phi_i\exp\left[in\phi_i - iK(t_f)r_i\cos(\phi_i - \varphi)\right] = \exp(in\pi/2 + in\varphi)\int_{-\pi/2}^{3\pi/2}\exp\left[i\big(n\alpha + K(t_f)r_i\sin\alpha\big)\right]d\alpha, \tag{14}$$

where $\varphi = \phi_f - \theta(t_f,t_0)$. Therefore, using the integral representation of the Bessel function [19]:

$$J_n(\xi) = \frac{1}{2\pi}\int_a^{2\pi+a}\exp\left[i(nx - \xi\sin x)\right]dx, \tag{15}$$

Eq. (14) can be written as

$$I = 2\pi(-1)^n\exp\big(in\pi/2 + in\phi_f - in\theta(t_f,t_0)\big)J_n\big(K(t_f)r_i\big). \tag{16}$$

Thus, we finally obtain for Eq. (11) the following expression

$$\eta_n(\mathbf{r}_{f\perp},t_f,\mathbf{r}_{i\perp},t_0) = s_n(t_f)\int_0^\infty dr_i\,r_i\exp\left[iA(t_f)r_i^2\right]R_n(r_i)J_n\big(K(t_f)r_i\big), \tag{17}$$

where

$$s_n(t_f) = \sqrt{2\pi}(-1)^n\exp\big(in\big(\pi/2 + \phi_f - \theta(t_f,t_0)\big)\big). \tag{18}$$

Since the Hankel transform of order n is defined as [19]:

$$\mathfrak{F}_n\left[g(r)\right] = \int_0^\infty dr\,rg(r)J_n(kr) \tag{19}$$

we can cast Eq. (10) in the form:

$$\psi_n(\mathbf{r}_{f\perp}, t_f) = \frac{m}{\sqrt{2\pi i\hbar\mu_1(t_f)}}\exp\left(i\frac{m\dot{\mu}_1(t_f)\mathbf{r}^2_{f\perp}}{2\hbar\mu_1(t_f)}\right) \times (-1)^n \exp\bigl(in(\phi_f - \theta(t_f, t_0) + \pi/2)\bigr)\mathfrak{F}_n\bigl[g(r_i)\bigr], \quad (20)$$

where $g(r_i)$ is given by

$$g(r_i) = \exp\bigl(i\bigl(m\mu_2(t_f)/2\hbar\mu_1(t_f)\bigr)r_i^2\bigr)R_n(r_i) \quad (21)$$

and where the k value that must be used in the Hankel transform of Eq. (19) is the quantity $K(t_f)$ defined by Eq. (13); its sign therefore is the same as the sign of $\mu_1(t_f)$. Eq. (20) describes the temporal evolution of the transversal motion in a uniform magnetic field that has an arbitrary time-dependence for $t > t_0$.

Let us now determine an explicit form of the time-evolved transversal wave function when the initial state is a Landau state of a particle in a static magnetic field B_0. Since the solution of Eq. (9) can be expressed by [18]:

$$R_n(r) = \gamma^{|n/2|}r^{|n|}\exp\bigl(-\gamma r^2/2\bigr)\sum_l a_{n,l}r^l, \quad (22)$$

where $a_{n,l}$ are the coefficients in the series expansion of the generalized Laguerre functions, our problem is reduced to the calculation of the Hankel transform of functions of the form $\exp(-\xi r^2)r^s$, where ξ is a complex variable. The result is:

$$\mathfrak{F}_n\bigl[\exp\bigl(-\xi r^2\bigr)r^s\bigr] = \frac{\Gamma((2+n+s)/2)}{2^{n+1}n!}\xi^{-(2+n+s)/2}k^n \times {}_1F_1\bigl((2+n+s)/2, 1+n, -k^2/4\xi\bigr), \quad (23)$$

where $\Gamma(x)$ is the gamma function, ${}_1F_1(a, b, c)$ is the confluent hypergeometric function and in our case $k = K(t_f)$, $\xi = \gamma - iA(t_f)$ and $s = |n| + l$. The expressions given above depend on the restriction $\mathrm{Re}[\xi] > 0$. From Eqs. (17), (19), and (23), we finally obtain:

$$\psi_n(\mathbf{r}_{f\perp}, t_f) = \frac{me^{(im\dot{\mu}_1(t_f)/2\hbar\mu_1(t_f))\mathbf{r}^2_{f\perp}}}{2\pi i\hbar\mu_1(t_f)} \times \sum_l a_{n,l}s_n(t_f)\frac{\gamma^{|n|/2}\Gamma((2+n+|n|+l)/2)K(t_f)^{|n|}}{2^{n+1}|n|!(\gamma - iA(t_f))^{(2+n+|n|+l)/2}} \times {}_1F_1\bigl((2+n+|n|+l)/2, 1+|n|, -K(t_f)^2/\bigl(4\bigl(\gamma - iA(t_f)\bigr)\bigr)\bigr). \quad (24)$$

Eq. (24) is the general expression for the time-evolved transversal wave function.

3. Caustics

In this section we will consider the case in which the magnetic field $B(t)$ is such that there exists a particular instant $t_f = t_f^0$ at which $\mu_1(t_f^0) = 0$. Under this condition, Eq. (5) for the propagator presents a singularity, and a more careful analysis must be performed.

From the theory of ordinary differential equations we know that the Wronskian of the solutions of Eq. (7), $W = \mu_1(t)\dot{\mu}_2(t) - \mu_2(t)\dot{\mu}_1(t)$, takes a constant value. By using the initial conditions for $\mu_1(t)$ and $\mu_2(t)$ it follows that at $t_f = t_f^0$ the functions $\mu_1(t)$ and $\mu_2(t)$ satisfy $\dot{\mu}_1(t_f^0)\mu_2(t_f^0) = 1$. For t_f close to t_f^0 we can approximate $\mu_1(t_f)$ by the first nonvanishing term in its Taylor series expansion around t_f^0, that is $\mu_1(t_f) \approx t_f - t_f^0$. Then, by using a standard representation of the Dirac delta function

$$\left(\frac{m}{2\pi i\hbar(t_f - t_f^0)}\right)\exp\left[\frac{im(\mathbf{r}_{f\perp} - \mathbf{r}_{i\perp})^2}{2\hbar(t_f - t_f^0)}\right] \to \delta(\mathbf{r}_{f\perp} - \mathbf{r}_{i\perp}) \quad \text{as } t_f \to t_f^0, \quad (25)$$

we obtain the following expression for the limiting value of the transversal propagator when $t_f \to t_f^0$

$$G_\perp\bigl(\mathbf{r}_{f\perp}, t_f^0, \mathbf{r}_{i\perp}, t_0\bigr) = \mu_2\bigl(t_f^0\bigr)\exp\left\{i\frac{m\dot{\mu}_2(t_f^0)\mu_2(t_f^0)r^2_{i\perp}}{2\hbar}\right\} \times \delta\bigl[\mathbf{r}_{f\perp} - \mu_2\bigl(t_f^0\bigr)\mathbb{R}\bigl(\theta\bigl(t_f^0, t_0\bigr)\bigr)\mathbf{r}_{i\perp}\bigr]. \quad (26)$$

That is, if the wave function at the initial time is highly localized around $\mathbf{r}_{i\perp}$, the wave function evolves to a final state that at time t_f^0 is also highly localized around the point obtained from $\mathbf{r}_{i\perp}$ by a rotation in the angle $\theta(t_f^0, t_0)$ and a scaling factor $\mu_2(t_f^0)$. The evolution of the transversal wave function is therefore:

$$\psi\bigl(\mathbf{r}_{f\perp}, t_f^0\bigr) = \mathrm{sgn}\bigl[\mu_2\bigl(t_f^0\bigr)\bigr]\exp\left\{i\frac{m\dot{\mu}_2(t_f^0)\mathbf{r}^2_{f\perp}}{2\hbar\mu_2(t_f^0)}\right\} \times \psi_0\left(\frac{1}{\mu_2(t_f^0)}\mathbb{R}\bigl(-\theta\bigl(t_f^0, t_0\bigr)\bigr)\mathbf{r}_{f\perp}\right), \quad (27)$$

showing that in the time evolution of the transversal wave function, an explicit time-dependent phase factor appears. Furthermore, if $\mu_2(t_f^0)$ remains bounded for all t in the range $[t_0, t_f^0]$ the resulting wave function will experience a finite deformation at these times, implying that the electron will remain localized.

Writing the final transverse position vector in phasorial language, $\mathbf{r}_{f\perp} \stackrel{\wedge}{=} r_f \exp\{i\phi_f\}$, we obtain

$$\psi_n\bigl(\mathbf{r}_{\perp f}, t_f^0\bigr) = \mathrm{sgn}\bigl[\mu_2\bigl(t_f^0\bigr)\bigr]\exp\left\{i\frac{m\dot{\mu}_2(t_f^0)\mathbf{r}^2_{f\perp}}{2\hbar\mu_2(t_f^0)}\right\} \times R_n\left(\frac{r_f}{\mu_2(t_f^0)}\right)\exp\bigl(in\bigl(\phi_f - \theta\bigl(t_f^0, t_0\bigr)\bigr)\bigr). \quad (28)$$

If in addition we have $\mu_2(t_f^0) \equiv 1$ and $\theta(t_f^0, t_0) = 0$, Eq. (28) is reduced to

$$\psi_n\bigl(\mathbf{r}_{\perp f}, t_f^0\bigr) = \exp\bigl\{im\dot{\mu}_2\bigl(t_f^0\bigr)\mathbf{r}^2_\perp/2\hbar\bigr\}\psi_0\bigl(\mathbf{r}_\perp, t_f^0\bigr). \quad (29)$$

Eq. (29) shows that in this case the time-evolved wave function differs from the initial wave function only in a phase factor. Our analysis does not depend on an adiabatic hypothesis, since the time-dependence of the magnetic field is arbitrary.

It is known in one-dimensional systems that a consequence of the time-dependence in the parameters is the existence of caustics (see for instance Ref. [20] for the case of the 1D harmonic oscillator). Our work shows how this behavior appears in the two-dimensional case as a consequence of the time-dependence of the magnetic field.

4. Conclusions

The dynamics of electrons in a uniform time-dependent magnetic field has been analyzed. An exact analytical solution for the time-evolved wave function has been obtained when the initial state corresponds to a superposition of Landau levels, and the dynamical phase factor that appears due to the time-dependence of the magnetic field has been explicitly derived, showing the existence of caustics in this two-dimensional case.

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References

[1] Y. Aharonov, D. Bohm, Phys. Rev. 115 (1959) 485.
[2] R.A. Webb, S. Washburn, C.P. Umbach, R.B. Laibowitz, Phys. Rev. Lett. 54 (1985) 2696.
[3] G. Timp, A.M. Chang, J.E. Cunningham, T.Y. Chang, P. Mankiewich, R. Behringer, R.E. Howard, Phys. Rev. Lett. 58 (1987) 2814.
[4] W.G. van der Wiel, Yu.V. Nazarov, S. De Franceschi, T. Fujisawa, J.M. Elzerman, E.W.G.M. Huizeling, S. Tarucha, L.P. Kouwenhoven, Phys. Rev. B 67 (2003) 033307.
[5] H. Ehrenreich, D. Turnbull (Eds.), Solid State Physics: Advances in Research and Applications, vol. 44, Academic Press, San Diego, 1991.
[6] T.A. Kennedy, R. Wagner, B. McCombe, D. Tsui, Phys. Rev. Lett. 35 (1975) 1031.
[7] E.A. Shaner, S.A. Lyon, Phys. Rev. B 66 (2002) 041402.
[8] A.A. Bykov, G.M. Gusev, J.R. Leite, A.K. Bakarov, A.V. Goran, V.M. Kudryashev, A.I. Toropov, Phys. Rev. B 65 (2001) 035302.
[9] M. Calvo, P. Lazcano, Optik 113 (2002) 31.
[10] M. Calvo, Optik 113 (2002) 233.
[11] M. Calvo, D. Laroze, Optik 113 (2002) 429.
[12] M. Calvo, Ultramicroscopy 99 (2004) 179.
[13] M.V. Berry, Proc. R. Soc. London A 392 (1984) 45.
[14] J.Y. Zeng, Y.A. Lei, Phys. Rev. A 51 (1995) 4415.
[15] D.Y. Song, Phys. Rev. A 61 (2000) 024102.
[16] X.B. Wang, L.C. Kwek, C.H. Oh, Phys. Rev. A 62 (2000) 032105.
[17] R. Feynman, H. Hibbs, Quantum Mechanics and Path Integrals, McGraw–Hill, New York, 1965.
[18] F. Constantinescu, E. Magyari, Problems in Quantum Mechanics, Franklin Book Company, 1971.
[19] J.W. Miles, Integral Transforms in Applied Mathematics, Cambridge Univ. Press, Cambridge, 1971.
[20] C. Grosche, F. Steiner, Handbook of Feynman Path Integrals, Springer-Verlag, New York, 1998.