



## Plaquette distributions for $\pm J$ Ising lattices<sup>☆</sup>

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### Abstract

Physical magnitudes of  $\pm J$  Ising systems are determined by their topological properties which depend on the number of frustrated plaquettes and on the distribution of curved plaquettes through the lattice. In the present paper, we consider two-dimensional lattices (3 homogeneous and 3 mixed lattices) and polyhedra (5 regular and 5 semi-regular). For small systems 3 different methods are used to find the abundance of curved plaquettes as function of  $x$  the concentration of ferromagnetic interactions. Coincidence among results is very good. In this way the function giving the probability for curved plaquettes is informed for 16 different systems. An application to obtain ground state energy in some cases is done at the end.

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### 1. Introduction

The Edwards–Anderson (EA) model [1] has been used as a simple way to approach some properties of spin glasses. Beyond that this model has inspired works in other fields dealing with frustrated systems [2,3]. In particular, the role played by frustrated plaquettes and the role of topology was brought up quite early [4–6]. It is precisely along this line of thought that we aim here to find probability functions of finding frustrated plaquettes for four families of two-dimensional lattices; a total of 16 lattices

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Table 1

Properties of the systems under consideration: number and kind of plaquettes (tP: triangular, sP: square, pP: pentagonal, hP: hexagonal); C: number of nearest neighbors (Coordination number); B: number of bonds

System	tP	sP	pP	hP	C	B	A	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$
Triangular TL ( $3^6$ )	1	0	0	0	6	3	1	1	-3	6	-4	0	0	0
Square SL ( $4^4$ )	0	1	0	0	4	4	1	0	4	-12	16	-8	0	0
Hexagonal HL ( $6^3$ )	0	0	0	1	3	6	1	0	6	-30	80	-120	96	-32
Kagomé KL (3,6,3,6)	2	0	0	1	4	10	$\frac{2}{3}$	1	0	-9	36	-60	48	-16
Five-p. FL ( $3^2, 4, 3, 4$ )	4	2	0	0	5	14	$\frac{2}{3}$	1	-1	0	4	-4	0	0
No-name NL (3,4,6,4)	2	3	0	1	4	17	$\frac{1}{3}$	1	6	-27	60	-72	48	-16
Tetrahedron T	4	0	0	0	3	6	1	1	-3	6	-4	0	0	0
Octahedron O	8	0	0	0	4	12	1	1	-3	6	-4	0	0	0
Hexahedron II	0	6	0	0	3	12	1	0	4	-12	16	-8	0	0
Icosahedron I	20	0	0	0	5	30	1	1	-3	6	-4	0	0	0
Dodecahedron D	0	0	12	0	3	30	1	-5	20	-40	40	-16	0	0
Tr. Tetrahedron TT	4	0	0	4	3	18	$\frac{1}{2}$	1	3	-24	76	-120	96	-32
Tr. Hexahedron TH	8	6	0	0	4	24	$\frac{4}{7}$	1	-3	8	-6	0	0	0
Tr. Octahedron TO	0	6	0	8	3	36	$\frac{4}{7}$	0	9	-39	92	-126	96	-32
Tr. Dodecahedron TD	20	0	12	0	4	60	$\frac{1}{4}$	4	-15	45	-70	60	-24	0
Tr. Icosahedron TI	0	0	12	20	3	90	$\frac{1}{8}$	3	15	-90	280	-480	432	-160

Numbers in parentheses represent the notation introduced by Grünbaum and Shephard for Archimedean lattices. Coefficients  $A$  and  $a_i$  refer to the polynomial expansions defined by Eq. (2).

are covered. With the knowledge of these functions, values of physical parameters associated to each lattice are possible as illustrated at the end of the paper.

Nearest neighboring sites on the lattice are joined by means of bonds which can be  $-J$  (ferromagnetic type: F) or  $+J$  (antiferromagnetic type: AF). The minimal circuit closing a loop over bonds is called plaquette. A plaquette is frustrated when it contains an odd number of AF bonds [7]. We will vary the concentration  $x$  of F bonds ( $1 - x$  for AF bonds) between 0.0 and 1.0 which alters the probability of having a frustrated plaquette.

Geometrical properties of the systems are summarized in Table 1, whenever possible we make use of the notation of Grünbaum and Shephard [8]. They can be grouped in four families in the following way:

- (a) Homogeneous Archimedean lattices having just one kind of plaquette and closing by means of periodic boundary conditions (PBC): ( $3^6$ ) triangular lattice (TL); ( $4^4$ ) square lattice (SL); ( $6^3$ ) hexagonal lattice (HL).
- (b) Mixed Archimedean lattices having more than one kind of plaquette and closing by means of PBC: (3,6,3,6) Kagomé lattice (KL); ( $3^2, 4, 3, 4$ ) five-points star lattice (FL); (3,4,6,4) (NL). (NL stays for no-name available for this lattice).

- (c) Regular polyhedra, which are closed lattices on the surface of regular polyhedra, having just one kind of plaquette: tetrahedron (T), octahedron (O), cube or hexahedron (H), icosahedron (I), and dodecahedron (D).
- (d) Semi-regular polyhedra, which are closed lattices on the surface of truncated regular polyhedra leaving plaquettes that are regular polygons of more than one kind: truncated tetrahedron (TT), cuboctahedron or truncated hexahedron (TH), truncated octahedron (TO), icosidodecahedron or truncated dodecahedron (TD), and spherical fullerene or truncated icosahedron (TI).

## 2. Analysis and discussion

For any given plaquette and a concentration  $x$  there is a probability that such plaquette is curved. Homogeneous lattices are shown in Fig. 1. To illustrate the method of exact enumeration of bonds (EEB) we apply it to TL. There are two ways of getting a frustrated triangular plaquette: (a) 2 F bonds and 1 AF bond with probability  $3x^2(1-x)$  (factor three comes for the three possible positions for the AF bond); (b) 3 AF bonds with probability  $(1-x)^3$ . Then, the probability of having a curved plaquette is the sum of these two

$$P_c^{TL}(x) = (1-x)(1-2x+4x^2) = 1-3x+6x^2-4x^3. \quad (1)$$

In an analogous way expressions  $P_c^{SL}(x)$  and  $P_c^{HL}(x)$  can also be found [9]. These and other probabilities are summarized on the right-hand side of Table 1 making use of

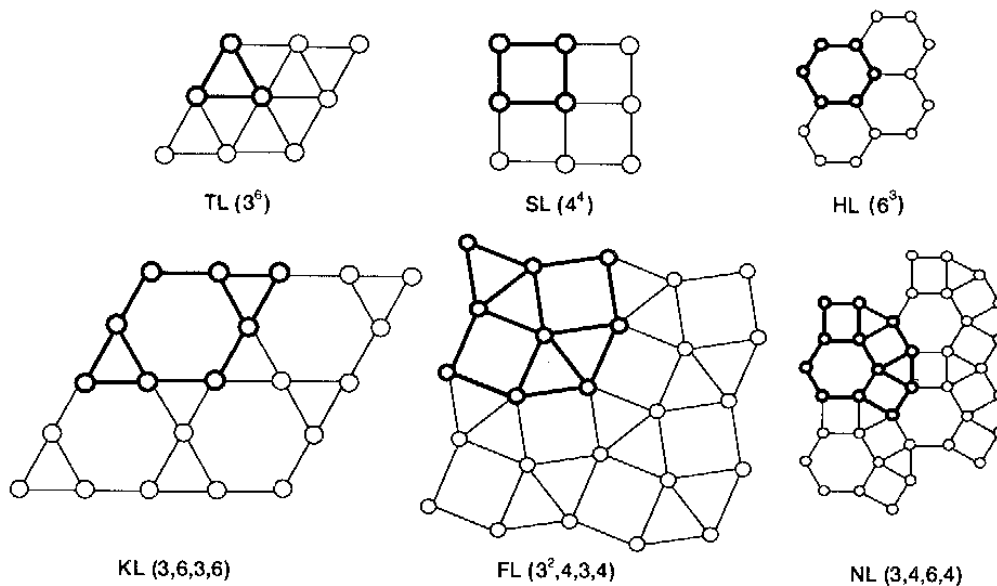


Fig. 1. Three homogeneous and three of the mixed Archimedean lattices. In this case the cell is highlighted by means of thick lines.

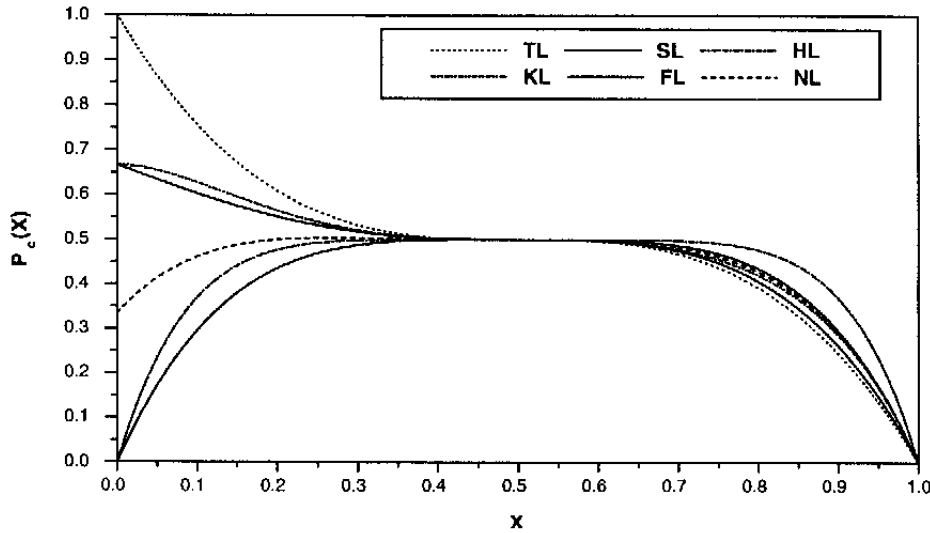


Fig. 2. Distribution of frustrated plaquettes for lattices of Fig. 1.

the general expression for these polynomials

$$P_c(x) = A \sum_{i=0}^n a_i x^i . \tag{2}$$

For each of the mixed lattices it is necessary to define a cell having translational property and each plaquette in the right proportion through the lattice as shown in the lower part of Fig. 1. Two analytical methods can be used: (a) EEB; (b) weighted probability according to the presence of each kind of plaquette (WP). The first method was first used for KL and FL [10,11], and later for NL [12]. We use in this paper the second method for the first time making use of the cell definition given in the lower part of Fig. 1 and the probability functions for individual plaquettes already reported above:

$$P_c^{KL}(x) = \frac{2}{3} P_c^{TL}(x) + \frac{1}{3} P_c^{HL}(x); \quad P_c^{FL}(x) = \frac{4}{6} P_c^{TL}(x) + \frac{2}{6} P_c^{SL}(x) , \tag{3}$$

$$P_c^{NL}(x) = \frac{2}{6} P_c^{TL}(x) + \frac{3}{6} P_c^{SL}(x) + \frac{1}{6} P_c^{HL}(x) . \tag{4}$$

The three corresponding polynomials are found in Table 1 and they are identical to those obtained by EEB [10–12]. The advantages of the method WP is evident from the point of view of simplicity.

Results for the 6 Archimedean lattices considered here are summarized in Fig. 2 for comparison purposes. Additionally, extensive numeric calculations agree well with these analytic results [9,11,12].

In the case of polyhedra no similar results are known. The whole polyhedron is considered as a cell for the application of the method EEB. Then in the cases

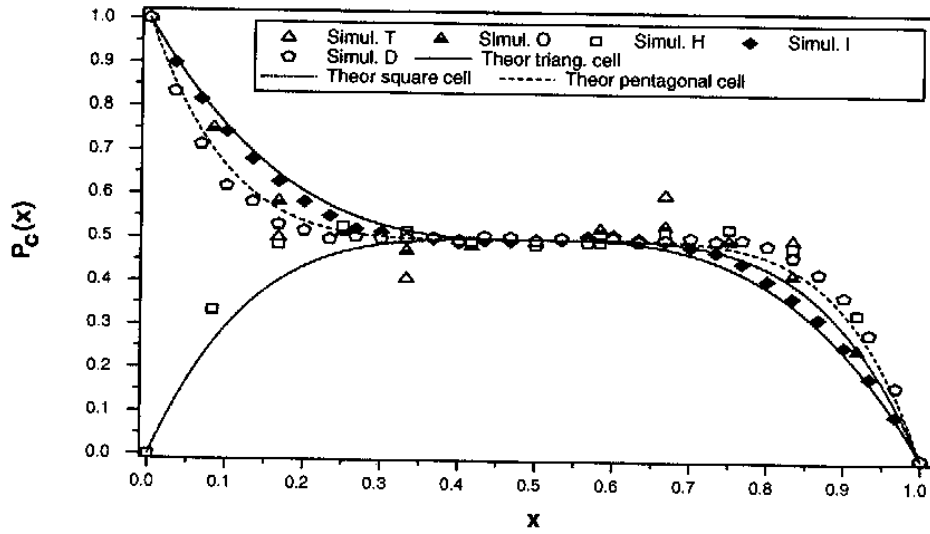


Fig. 3. Distributions of frustrated plaquettes for the regular polyhedra. Lines (symbols) correspond to analytic expressions (numerical results).

of T, O and I we recover  $P_c^{TL}(x)$  the probability corresponding to a triangular frustrated plaquette. In the case of H, we get the probability for a square frustrated plaquette  $P_c^{SL}(x)$ . In the case of D we get a function which is exactly the same to the probability of having a pentagonal frustrated plaquette, as given in Table 1.

The continuous curves in Fig. 3 represent functions  $P_c^{TL}(x)$ ,  $P_c^{SL}(x)$ , and  $P_c^{PL}(x)$ . The symbols in Fig. 3 give average results over 1000 random samples for each polyhedron at different concentration values  $x$  [13]. There is a good agreement between simulations and theoretical expressions especially when the number of bonds in the polyhedra increases.

Previous procedure is now extended to semi-regular polyhedra. Probability functions are given in Table 1 and they are plotted in Fig. 4 along with numerical simulations (based on 1000 random samples for each case) for two of the systems for comparison purposes.

To illustrate the application of these results we will tackle two problems: (a) ground state energy of the homogeneous lattices at  $x = 0.5$ ; (b) ground state energy of all systems at  $x = 0.0$  where the differences among systems are at the apex. Let us start from the general property

$$\varepsilon_g = -1 + \frac{P}{B} \langle \lambda_g \rangle P_c, \quad (5)$$

where  $P$  is the total number of plaquettes in the system,  $\langle \lambda_g \rangle$  is the ground average frustration segment for the lattice and  $P_c$  is the probability function of having a frustrated plaquette. The value of energy is determined by average frustration segment  $\langle \lambda_g \rangle$ , which can be expressed as an expansion in the form

$$\langle \lambda_g \rangle = \sum_{\ell=1}^{\infty} \ell \times p_{\ell}, \quad (6)$$

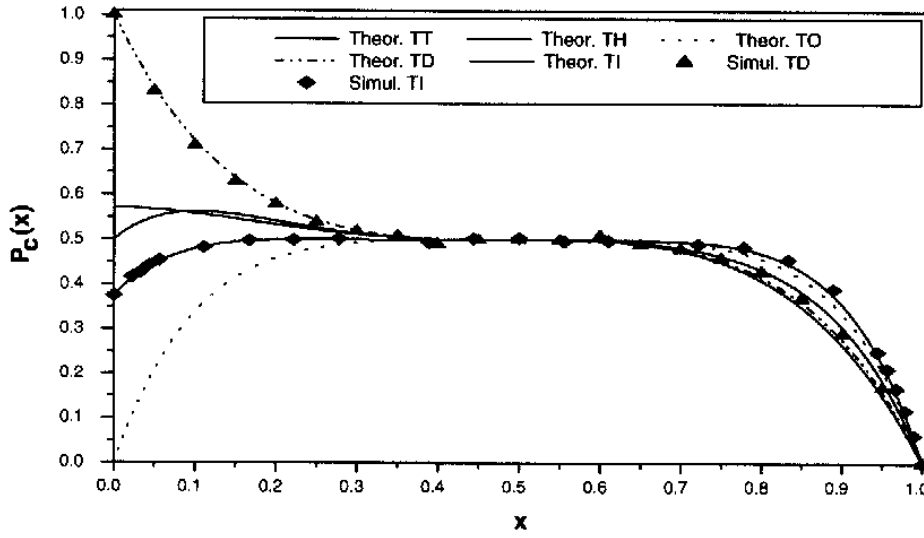


Fig. 4. Distributions of frustrated plaquettes for semi-regular polyhedra. Lines (symbols) correspond to analytic expressions (numerical results).

Table 2  
Factors entering Eq. (5) and energy per bond  $\epsilon_g(0.0)$  associated to the 16 systems illustrated in Table 1

System	TL	SL	HL	KL	FL	NL	T	O	H	I	D	TF	TH	TO	TD	TI
P/B	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{6}{15}$	$\frac{4}{9}$	$\frac{7}{12}$	$\frac{7}{18}$	$\frac{8}{15}$	$\frac{16}{45}$
$\langle \lambda_g \rangle$	1	0	0	2	1	2	1	1	0	1	1	2	2	0	$\frac{5}{4}$	2
$P_c$	1	0	0	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	1	1	0	1	1	$\frac{1}{2}$	$\frac{4}{7}$	0	1	$\frac{3}{8}$
$\epsilon_g$	$-\frac{1}{3}$	-1	-1	$-\frac{1}{3}$	$-\frac{3}{5}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	-1	$-\frac{1}{3}$	$-\frac{3}{5}$	$-\frac{5}{9}$	$-\frac{1}{3}$	-1	$-\frac{1}{3}$	$-\frac{11}{15}$
Ref.	9	9	9	11	11		13	13	13	13	13					13

Last row gives the reference number leading to numerical simulations (when available) verifying this result.

where  $l$  is the corresponding length of the frustration segment having a probability  $p_l$  characteristic of each system.

- (a) After some topological considerations, simple algebra to truncate the series above up to  $l = 2$ , and the value  $P/B$  as given in Table 2, it is found that for the case  $x = 0.5$ ,  $\langle \lambda_g^{TL} \rangle \approx 1.3$ ,  $\langle \lambda_g^{SL} \rangle \approx 1.2$ , and  $\langle \lambda_g^{HL} \rangle \approx 1.1$ . Then, according to Eq. (5) we can get

$$\epsilon_g^{TL}(0.5) > \epsilon_g^{SL}(0.5) > \epsilon_g^{HL}(0.5). \quad (7)$$

- (b) For the case  $x = 0.0$  all plaquettes with odd number of bonds frustrate. Thus, for systems with only hexagonal and/or square plaquettes there are no frustration segments and  $\langle \lambda_g \rangle = 0.0$ . For systems with only triangular or pentagonal plaquettes, similar frustrated plaquettes neighbor each other and  $\langle \lambda_g \rangle = 1.0$ . However, when

mixture of plaquette occur  $\langle \lambda_g \rangle$  is generally larger than 1.0 and even fractional values may occur. A summary of values of  $\langle \lambda_g(0.0) \rangle$  is presented in Table 2, where also the ground state energies  $\varepsilon_g(0.0)$ , after Eq. (5), are also given. These results agree with numerical results when available as indicated in the last row of Table 2.

### 3. Conclusions

Probability of having a frustrated plaquette was obtained for 16 different geometries as function of  $x$ , the concentration of ferromagnetic bonds. In the case of systems having just one kind of plaquette (homogeneous lattices and regular polyhedra) the probability obtained by just one plaquette or several plaquettes sharing edges (EEB on the whole polyhedra) yield the same functions. When there is more than one kind of plaquette method WP yields the same results as the method EEB with less effort. Numerical simulations on small systems tend to agree well with analytic results. This agreement is better as systems get larger and the number of bonds increases. Systems having only plaquettes formed by even number of bonds produce symmetric functions with respect to the point  $x = 0.5$ . Systems having at least one type of plaquette with odd number of elements produce asymmetric functions. All probability functions show a plateau with the result 0.5 within the approximate range  $0.4 \leq x \leq 0.6$ . For values of  $x$  over this range  $P_c(x)$  decreases towards 0.0 in slightly different ways according to the topology of the systems. On the other hand, large differences occur for values of  $x$  tending to 0.0, which is probably where a larger variety of behaviors can be expected. The case of polyhedra offers interesting features since it is not necessary to impose boundary conditions on them. Additionally, it is possible to observe the effects of the pentagonal plaquette for the cases of D, TD and TI. Once  $P_c(x)$  is known for any lattice, analytic expressions for physical parameters can be obtained as functions of  $x$ . The accuracy of this is limited by a truncated series in terms of frustration segments.

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